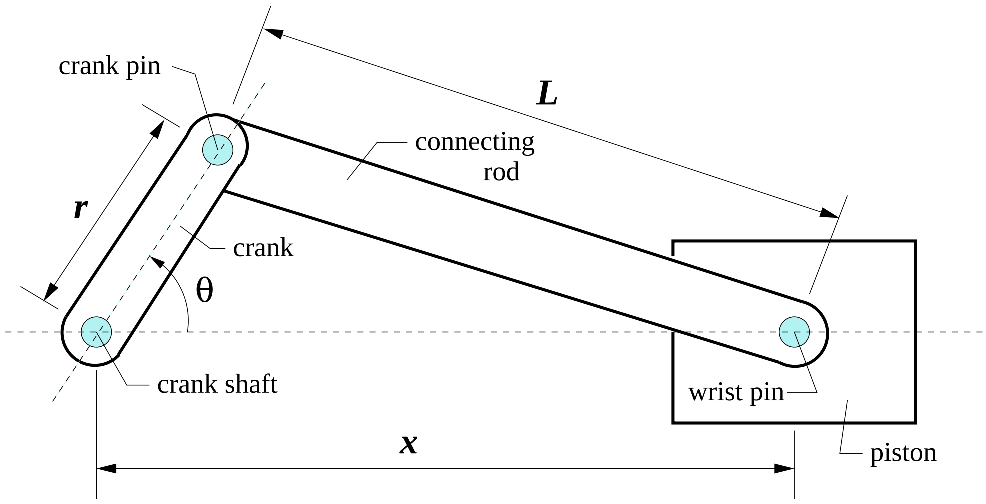
1. **Simple Crank**

Consider the mechanism illustrated. As the crank rotates around the fixed crank shaft, the piston moves back and forth along a horizontal axis. Assume the crank length $r = 7$ cm, the connecting rod length $L = 24$ cm and the crank is turning slowly round and round with a period of two seconds.

Assume $\theta = 0$ at time $t = 0$.



A visual representation can be seen at

http://www.brockeng.com/mechanism/index.htm

by clicking on "Simple Crank".

<u>Without calculus</u>

Let θ be the angle illustrated (in natural radians), t be the time (in seconds), and x ($x > 0$) the distance (in cm) of the piston from the crank shaft.

(a) What is the angle θ at a general time t?

(b) By inspection, what are the maximum and minimum values of x? What are the corresponding angles θ and times t when they occur?

(c) Consider the first instant when the crank happens to be perpendicular to the connecting rod. At this instant, use simple geometry to calculate the length x, as well as the values of $\sin\theta$ and $\cos\theta$. What is the corresponding angle θ and time t?

(d) Use the cosine rule to prove that

$$x^2 - 14x\cos\theta - 527 = 0\ .$$

What is the distance x when $t = \frac{1}{2}$ sec? When $t = \frac{1}{3}$ sec?
What are the first two values of the angle θ and corresponding times t when $x = 25$ cm? When $x = 18$ cm?

(e) Let ϕ be the angle that the connecting rod makes with the horizontal axis.
Using a geometric argument, what is the maximum value of ϕ, and when is the first time it occurs?

(f) Explain why

$$\frac{\sin\phi}{r} = \frac{\sin\theta}{L}.$$

What is the angle ϕ at the instant in part (c)? After $t = \frac{5}{6}$ sec?

With calculus

(g) Use the relationship $x^2 - 14x\,\cos\theta - 527 = 0$ and implicit differentiation with respect to θ to find $\frac{dx}{d\theta}$.

Hence determine the two values of θ, $0 \le \theta < 2\pi$ where the derivative is zero, and the corresponding values of $x > 0$. (**Hint:** 17x31=527.) With the aid of a sign diagram, find the maximum and minimum values of x. What does this confirm?

(h) Explain why $\frac{d\theta}{dt} = \pi$ radians/sec.

At the instant when the crank happens to be perpendicular to the connecting rod, calculate the velocity of the piston in cm/sec.

(i) Use

$$\frac{\sin\phi}{r} = \frac{\sin\theta}{L}$$

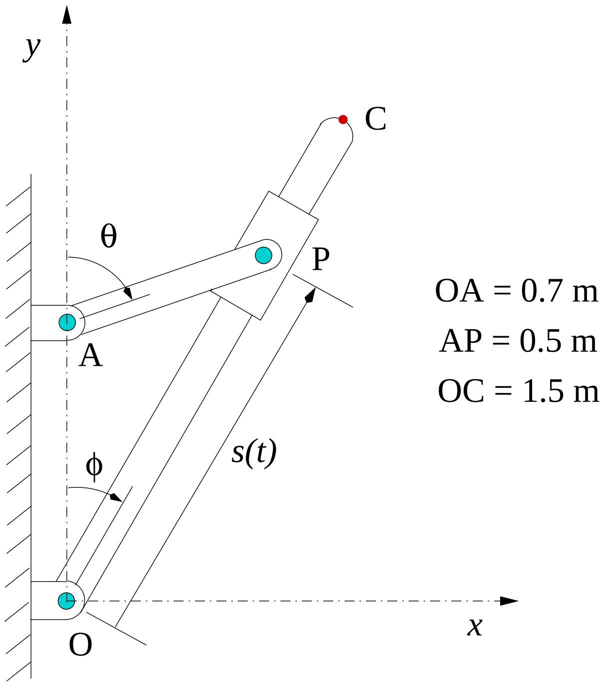
and implicit differentiation with respect to θ to confirm your answer in part (e) for the maximum value of ϕ.

Use implicit differentiation with respect to t (or use the chain rule of differentiation) to find the angular velocity (in radians/sec) of the angle ϕ at the instant in part (c).

With the aid of a computer or graphics calculator

(j) Let P be the mid-point of the connecting rod. Sketch the path followed by P during a single cycle. (**Hint:** express the coordinates of P in terms of θ, and then let θ vary from 0 to 2π. You should obtain an "egg-shape", well known to engineers.)

2. Consider the *Swing Arm Quick-Return Mechanism* illustrated, which is driven by the crank ("driving link") AP. The slider at P is free to slide along the "driven link" OC.



Assume that, at time $t = 0$, the crank and driven link are in the vertical position (so that $\theta = 0$, $\phi = 0$) and thereafter the crank AP rotates in a clockwise direction at 20 revolutions per minute about A.

A visual representation can be seen at

http://www.brockeng.com/mechanism/index.htm

by clicking on "Swing Arm Quick Return".

Without calculus

Let θ and ϕ be the angles illustrated (in natural radians), t be the time (in seconds), and $s(t)$ ($s(t) > 0$) the distance (in m) of the point P from the origin. In other words, $s(t)$ represents the linear motion of the slider up and down the driven link.

(a) What is the angle θ at a general time t?

(b) Use simple geometry to determine the maximum value (in radians) of the angle ϕ. At what time does it first occur?

Assuming the minimum value of ϕ is merely the negative of this (a mirror reflection on the other side), at what time does it first occur?

Hence, in a given cycle of 3 seconds, for what total amount of time is the driven link moving to the left, and how long is it moving to the right? What percentage of time is the driven link moving to the left, and to the right? Why do you think it is called a "Swing Arm Quick-Return Mechanism"?

(c) Prove that

$$\tan\phi = \frac{5\sin\theta}{7+5\cos\theta}\,.$$

Find both angles when $t = \frac{3}{4}$ sec, when $t = 1$ sec, when $t = 1\frac{1}{2}$ sec, and when $t = 1\frac{3}{4}$ sec.

(d) Check that the maximum value of the angle ϕ (refer to part (b)) is greater than $\frac{\pi}{4}$ radians (which is 45^o).
Hence the driven link OC must pass $\frac{\pi}{4}$ on the way out towards the maximum, and then again on the way back. Because you know that $\tan\frac{\pi}{4} = 1$, these two situations will occur when

$$1 = \frac{5\sin\theta}{7+5\cos\theta}\,.$$

Cross-multiply, square both sides, and then use the trigonometric identity $\sin^2\theta = 1-\cos^2\theta$ to prove that $\phi = \frac{\pi}{4}$ when $\cos\theta_1 = -\frac{3}{5}$ and when $\cos\theta_2 = -\frac{4}{5}$. Find θ_1 and θ_2, and the times t_1 and t_2 when they occur.

(e) By inspection, what are the maximum and minimum values of $s(t)$, and when do they first occur?

(f) Use geometry to prove that

$$s = 0.1\sqrt{74+70\cos\theta} \quad \text{(metres)}.$$

Knowing that $\cos\theta$ varies between +1 and -1, check your previous maximum and minimum values of s.

(g) What is s at the instant when ϕ is a maximum?

(h) Show that $s(t_2) = \frac{3}{4}\,s(t_1)$.

With calculus

(i) Differentiate the implicit relationship $\tan\phi = \dfrac{5\sin\theta}{7+5\cos\theta}$ with respect to θ to prove that $\dfrac{d\phi}{d\theta} = 0$ when $\cos\theta = -\frac{5}{7}$. Hence confirm the maximum value of ϕ found in part (b).

(j) Evaluate $\dfrac{d\phi}{d\theta} =$ at both $\theta = \theta_1$ and $\theta = \theta_2$. Use the chain rule

$$\frac{d\phi}{dt} = \frac{d\phi}{d\theta}\,\frac{d\theta}{dt}$$

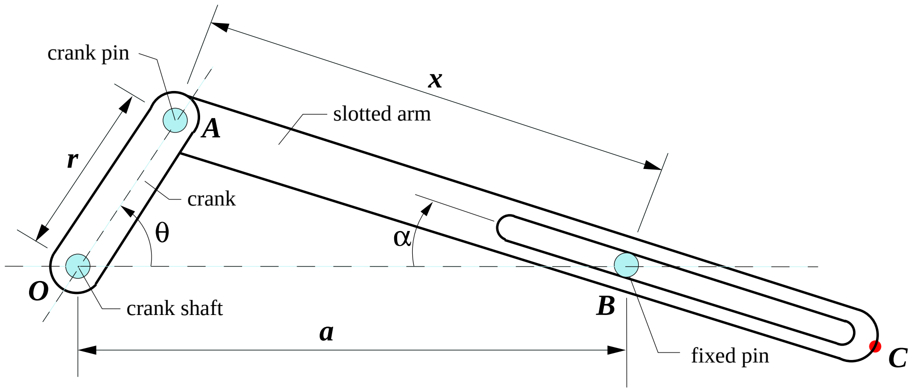
to find the angular velocity $\phi'(t)$ at each of these instances. Why do these values reinforce your concept of a "Swing Arm Quick-Return Mechanism"?

(k) The linear velocity of the slider along OC is given by

$$v'(t) = \frac{ds}{dt} = \frac{ds}{d\theta}\,\frac{d\theta}{dt}\,.$$

Find the velocity of the slider at both $t = t_1$ and $t = t_2$.

3. Consider the mechanical device illustrated.



The crank OA revolves counterclockwise about the fixed point O at two revolutions per second. The length OA is $r = 20$ cm. The driven link AC of length 85cm is a slotted arm constrained by the fixed pin B at a distance $a = 60$ cm from O.

<u>Without calculus</u>

At any given instant, let the angle of rotation AOB be denoted by θ, the corresponding angle ABO be denoted by α and the distance AB be denoted by x. Assume the angle $\theta = 0$ at time $t = 0$.

(a) What is the angle θ at a general time t?

(b) Use simple geometry to determine the maximum value (in radians) of the angle α. What is the corresponding value of θ, and at what time does it first occur?

(c) Prove that

$$\tan\alpha = \frac{\sin\theta}{3 - \cos\theta}\,.$$

Find both angles when $t = \frac{1}{12}$ sec, when $t = \frac{1}{8}$ sec, and when $t = \frac{1}{6}$ sec.

(d) By inspection, what are the minimum and maximum values of $x(t)$, and at what times do they first occur?

(e) Prove that $x = 20\sqrt{10 - 6\cos\theta}$. Hence find the length x when $t = \frac{1}{12}$ sec, when $t = \frac{1}{8}$ sec, when $t = \frac{1}{6}$ sec, and when the angle α is a maximum.

<u>With calculus</u>

(f) Differentiate the implicit relationship $\tan\alpha = \dfrac{\sin\theta}{3 - \cos\theta}$ with respect to θ to prove that $\dfrac{d\alpha}{d\theta} = 0$ when $\cos\theta = \frac{1}{3}$. Hence confirm the maximum value of α found in part (b).

(g) Use the chain rule

$$\frac{d\alpha}{dt} = \frac{d\alpha}{d\theta}\,\frac{d\theta}{dt}$$

to find $\alpha'(t)$ at each of the times $t = \frac{1}{12}$ sec, $t = \frac{1}{8}$ sec, and $t = \frac{1}{6}$ sec.

(h) Use the formula $x = 20\sqrt{10 - 6\cos\theta}$ to find $\dfrac{dx}{d\theta}$, and hence confirm your answers in part (d).

With the aid of a computer or graphics calculator

(i) Let C be the end of the slotted arm, as illustrated. Sketch the path followed by C during a single cycle. (**Hint:** express the coordinates of C in terms of θ and α. Then let θ vary from 0 to 2π. Use the arctan (i.e. $\tan^{-1}$) function to find the corresponding values of α from the relationship in part (c), and hence the coordinates of C. You should obtain a "sideways globule-shape", well known to engineers.)