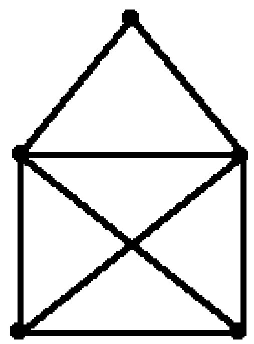
Year 11 Mathematics Awareness Workshops

Street Networks

(Notes prepared by Associate Professor David Panton University of South Australia)

Introduction

You may have been given this puzzle when you were in Primary School. The diagram below shows what looks like a house. The puzzle involves drawing the house by starting at the point S and finishing at the point E, without taking your pen off the page and without going over any edge more than once. Try it.



S E

Figure 1

Street Networks

Although the connection might be hard to see at the moment, this puzzle relates to some important applications of mathematics in our society, namely with street networks. Think of the 'points' on the above diagram as intersections and the 'edges' as streets joining them**.** We could equally think of the points as towns and the edges as roads joining the towns. There are many important tasks related to street networks, including postal deliveries, garbage collection, street sweeping, meter reading, snow ploughing and so on. Most of us would have been woken up early in the morning by a garbage truck driving down the street, once to collect the bins on one side of the road, and again to collect the bins on the other side of the road. Having dozed off again for the second time, you are only to be woken again by the truck going down your street a third time! Why has it done this? We will talk about garbage trucks again later, but first consider a very simple street network as shown below, where the point (***called a vertex or node***) in the top left hand corner is where the post-office is located. A postie must start at the post-office and return, having delivered letters along each street. The diagram below is an example of a ***network*** that consists of a set of ***vertices*** joined by ***edges***. It is a mathematical model that, as you will see, can be used to solve some useful and basic problems.

If we assume that all edges or streets are of length 1, what is the least distance the postie needs to travel? As you can see there are 12 streets, so ideally the postie should travel only a distance of 12, but is this possible?

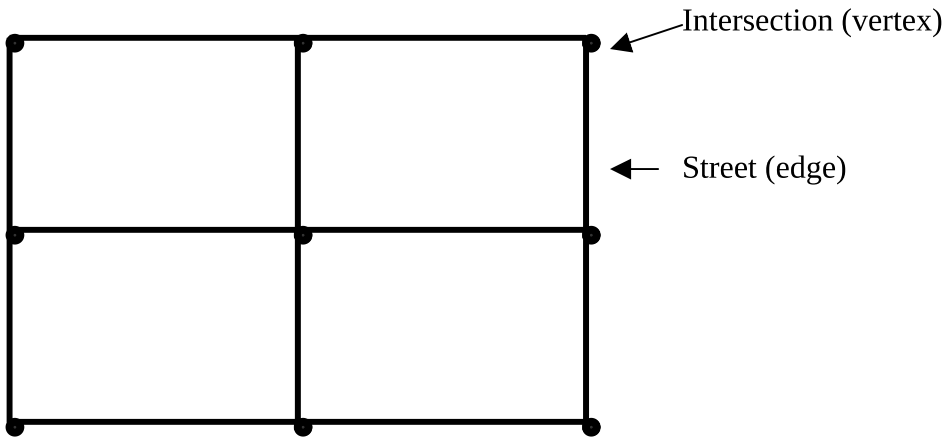
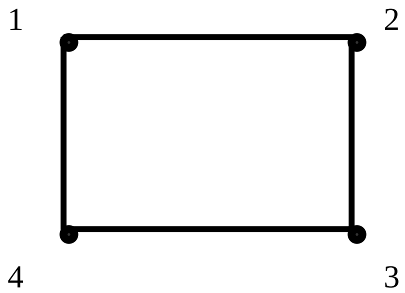


Figure 2

Now let's try something slightly easier:



Easy! Simply take the path

1 → 2 → 3 → 4 → 1

Figure 3

So what is it about this network that is fundamentally different from the other?

The Koenigsberg Bridge problem

The cycle from vertex 1 back to vertex 1 is an example of an ***euler tour*** named after the famous Swiss mathematician Leonhard Euler (pronounced 'oiler'). Euler was the first person to use a network model to solve problems of this sort. At the time (1700s), Euler had a friend who was mayor of the town of Koenigsberg (now called Kaliningrad in Russia). His friend wrote to him with the following problem:

The problem, which I understand is quite well known, is stated as follows: In the town of Koenigsberg in Prussia there is an island called Kneiphof, with two branches of the river Pregel flowing around it. There are 7 bridges - a, b, c, d, e, f, and g - crossing the two branches. The question is whether a person can plan a walk in such a way that he will cross each of these bridges once but not more than once. I was told that while some denied the possibility of doing this and others were in doubt, no one *maintained that it was actually possible.*

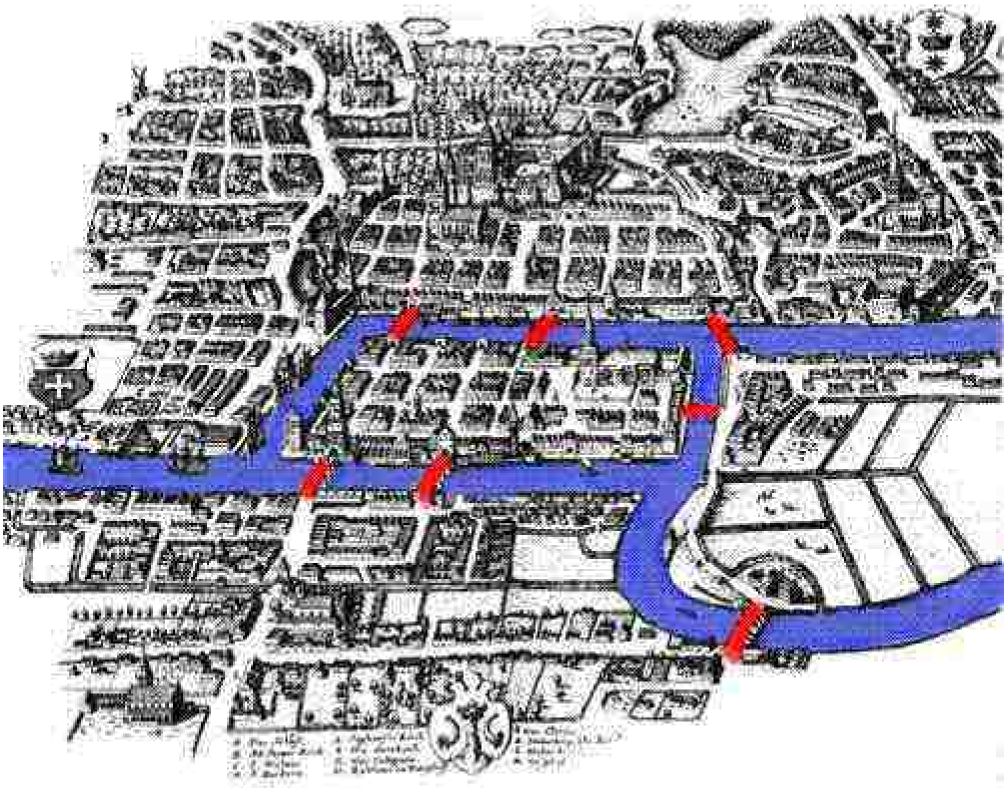


Figure 4

The picture above shows the river, its islands and the bridges. This is drawn schematically in the figure below.

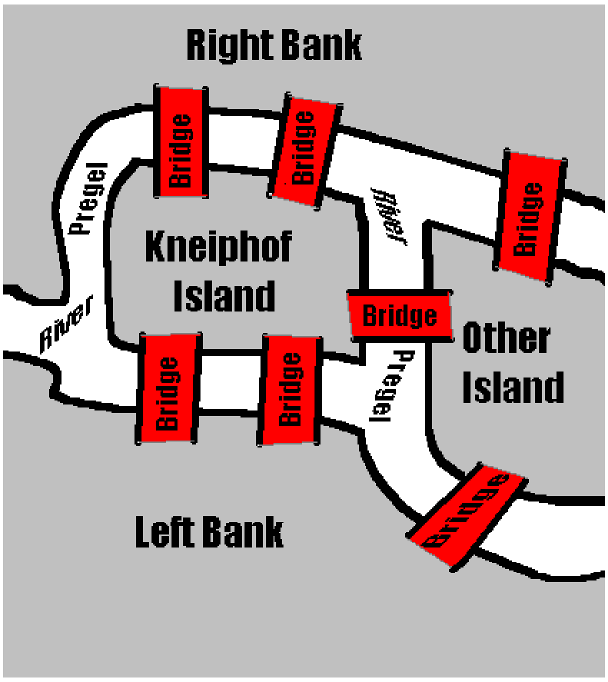


Figure 5

Euler came up with a theory that solved this problem. To understand Euler's solution to this problem we will reconstruct his steps. To solve this bridge problem Euler changed the picture above into a network. Euler simplified the picture by labelling the islands A and D and the shores of the river B and C. The bridges are represented by the edges joining these land masses. Check it out.

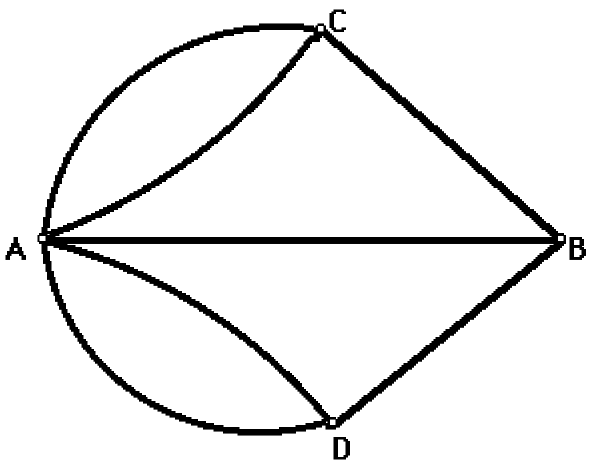


Figure 6

This was Euler’s mathematical model with which he made a simple observation and the answer to the problem. His simple observation relates to the number of edges attached to each vertex. This is called the ***degree of the vertex***. So, as you will see, vertex A has degree 5, while vertex B has degree 3. If we look at an isolated vertex with several edges attached:

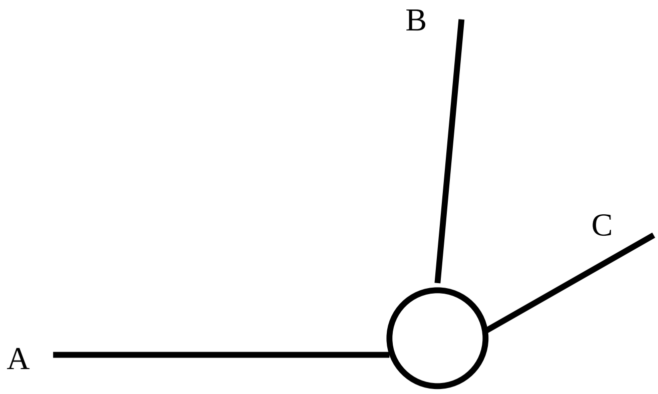


Figure 7

You can see that if you enter the vertex from edge A and leave by edge B, then at some stage you also have to either leave or enter via edge C. The only way you can do this is by going along either edge A or B again, which is disallowed. Euler's conclusion was that each vertex had to be associated with an ***even*** number if edges. Specifically, a necessary condition for an euler tour is that every vertex has to have an even degree, i.e. an even number of associated edges. Since the network associated with the Koenigsberg bridge problem has all of its vertices of odd degree there was no chance that the people of this city could achieve their task. Euler was also able to prove that as long as every vertex was of even degree such a tour could always be found in a network. In other words, even degree of all vertices was ***necessary and sufficient*** for an euler tour to exist.

What about our house puzzle at the beginning? The two vertices at S and E have odd degree. But in this case we were not required to return to S. In fact in this case we have found what is called an ***euler path***. That is we have traversed every edge exactly once but finished at a different vertex. In order to guarantee an euler path the network can have at most 2 vertices of odd degree.

The Postie problem

We can now see why the original street network in Figure 2 doesn't have a simple postie tour. Let's draw this network again and label the vertices. We now know that the postie can't deliver to all the houses without going more than a distance of 12. The postie will have to travel along some of the streets again. This is called ***dead-heading***. Figure 8 shows one possible solution where streets (2,5), (5,6), (4,5), and (5,8) are repeated, but only after the postie has already delivered down those streets. Can you do better than this? The total distance in this case is now 16. We call the process of adding the extra edges so that an euler tour is possible, ***eulerising the network***. Note now that every vertex has an even degree.

Look at the alternative eulerisation in Figure 9. Why can't we do this?

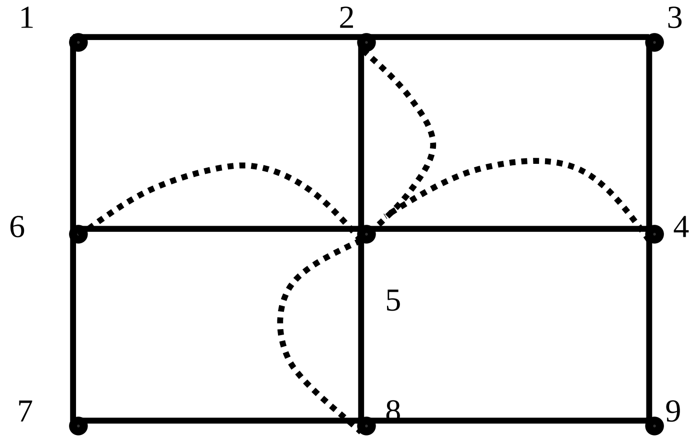


Figure 8

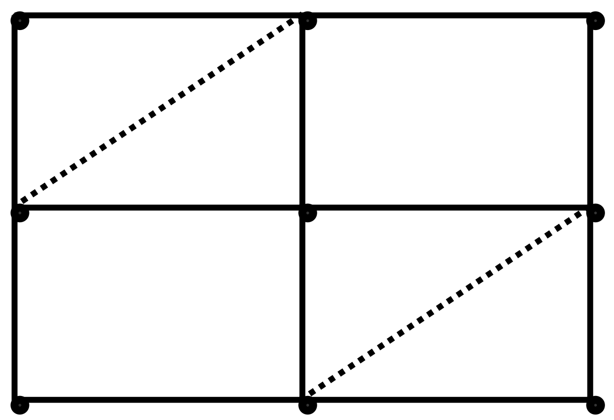
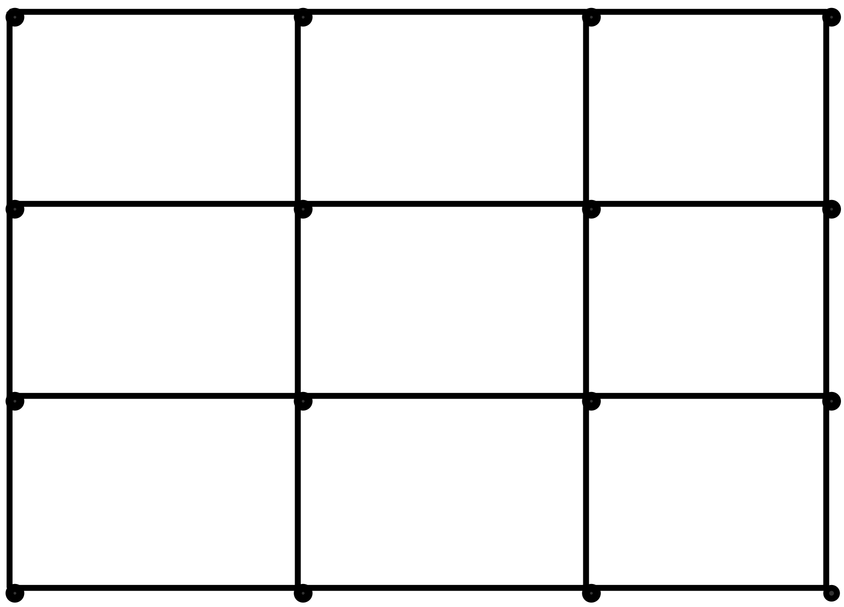
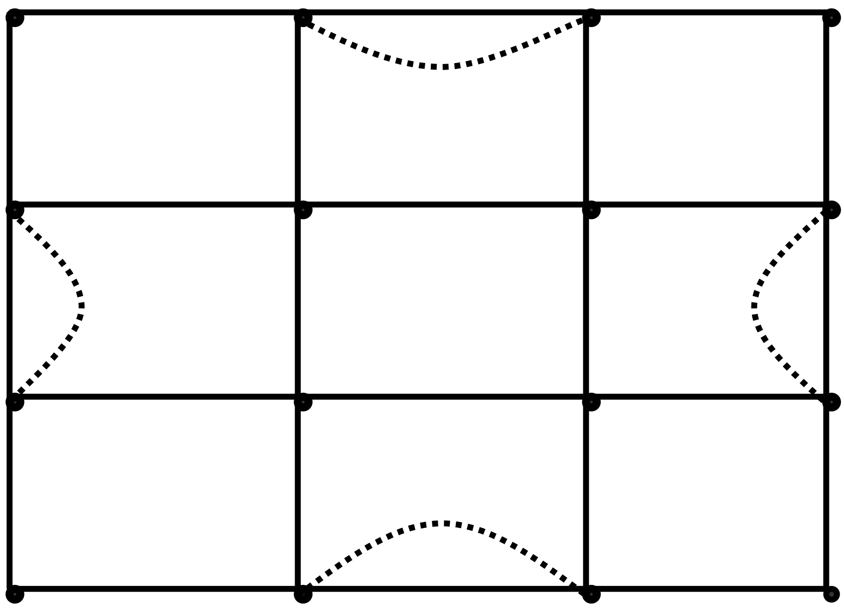


Figure 9

Exercise
Try eulerising the network shown below. Again assume that all streets are of the same length.



One possible solution might look like this



General street networks

What happens if the streets are not all the same length? Consider the example in Figure 10.

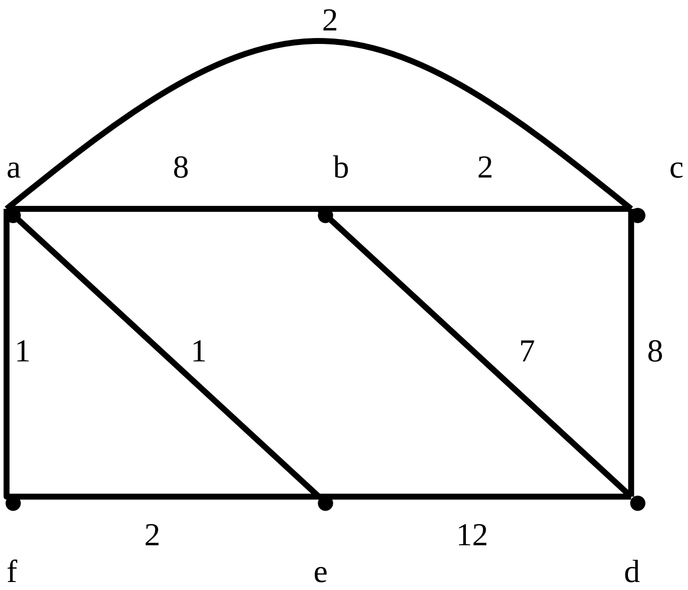


Figure 10

The length of each street is shown on each edge. Note that the length of an edge as it is drawn has nothing to do with the actual length of the street.

How do we eulerise the network in this case?

First of all notice that there are four vertices which have odd degree, namely b, c, d and e. If you look back at all the other examples of networks which did not have an euler tour you will notice that in every case there are an even number of vertices with odd degree. Is this purely coincidental? Well no. In fact it is not too hard to prove that in any network the number of vertices of odd degree is always even. Can you prove this?

This fact is very important in what follows. Remember we must add certain edges (simply repeat existing edges) so that every vertex has even degree. Now it is important that we take into account the fact that some edges (streets) are longer than others. In Figure 10, we certainly want to avoid adding edges (e,d), (c,d) or (b,d) for example. We know that the postie has to travel at least a distance of 43, but we want to keep the amount of dead-heading as small as possible. This is called an optimisation problem.

Here is an outline of how to solve this optimisation problem.

Step 1: Identify the vertices with odd degree nodes. There must be an even number of them. In this case we have vertices b, c, d and e.

Step 2: Construct another network using these vertices. Using the original network find the shortest distance between each pair of vertices for this new network. The results for step 2 are shown in Figure 11.

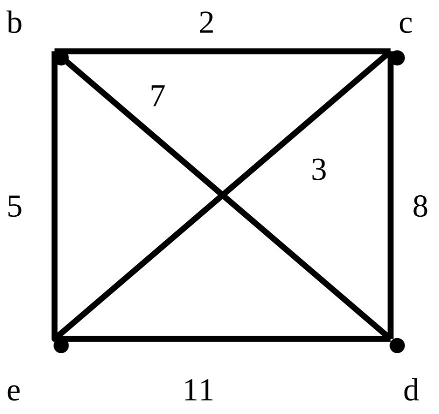


Figure 11

The numbers on each edge now represent the shortest distance between the vertices to which they are connected. So the shortest distance from b to c is 2, along the edge from b to c. However the shortest distance from e to d is 11 since it is shorter to go from e to d via vertex 1 than to go there directly. Notice then that some of the edges on this network represent more than one edge on the original network.

We now look for pairings of vertices in this new network. These are called **matchings**. There are three distinct matchings for this network, as shown in Figure 12. Note that we can only do this with an even number of vertices.

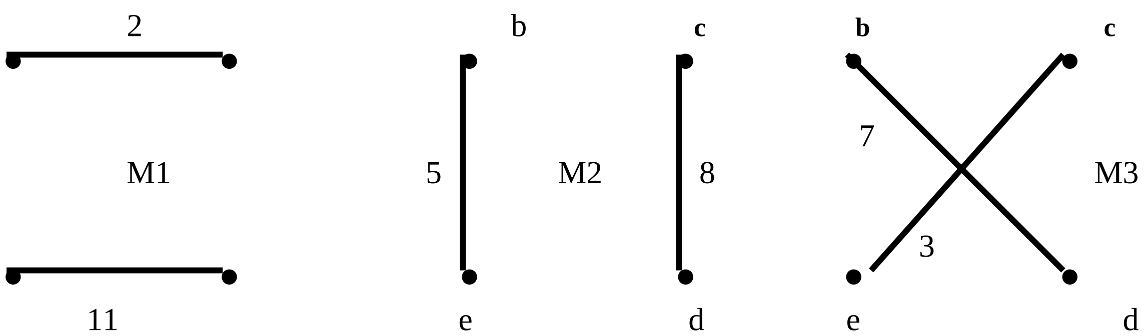


Figure 12

Every matching has a value associated with it. Thus M1 has a value of 13, M2 a value of 13 and M3 a value of 10. What do these matchings mean as far as our postie problem is concerned? Each matching represents a way of adding an edge (or a sequence of edges) to the network, in such a way that every odd degree vertex is rectified. Take M2 for example. The edge (b,e) represents the sequence of edges b, c, a, e of total length 5, while the edge (c,d) represents the single edge connecting c and

d. The length of each matching represents the ***extra distance*** which must be travelled, so obviously we want to take the matching which is the smallest, namely M3 of length 10.

Having decided that M3 is the best, let's go back to the network in Figure 10 and insert the edges related to M3.

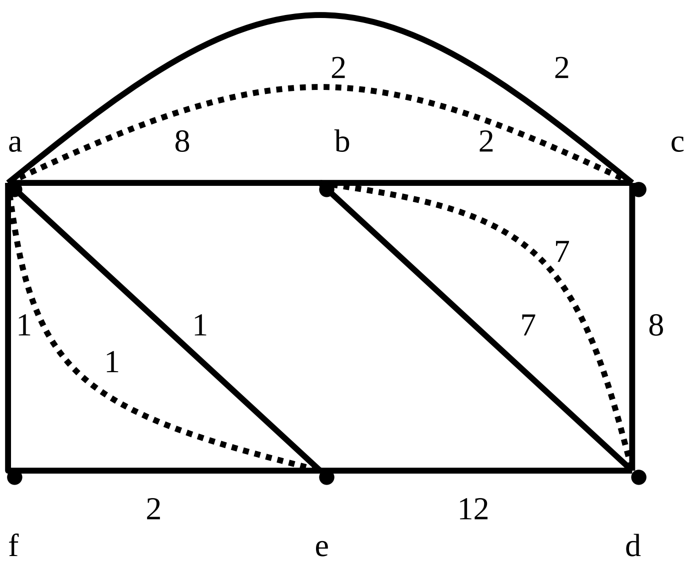


Figure 13

Voila! Every vertex now has even degree and the postie can now find an euler tour which visits each street exactly once with the minimum amount of dead-heading. In this case the extra distance travelled is 10.

The last word

In practice when you look for an euler tour in such a network you will find that there are many alternatives. For example starting at a, one possibility is

a, b, c, a, c, d, b, d, e, a, e, f, a   while another is   a, f, e, a, e, d, b, d, c, b, a, c, a

In each of these euler tours there are U-turns or right hand turns involved, For example in the first we have U-turns at intersections a, b and c as well as several right hand turns. While these might be acceptable in theory, in practice they are often not safe, especially for a large garbage truck at a major intersection. There are ways of constructing tours that can minimise U-turns and right hand turns.