3.8. Nonlinear Models and Linearization

Most natural systems are nonlinear. An important criterion that distinguished nonlinear systems from linear systems is the principle of superposition. If this principle holds good as it happens in linear systems, the sum of all the individual outputs due to several individual inputs, each considered to be acting alone on the system is equal to the output due to all the inputs acting simultaneously on a system. Nonlinear systems do not obey the principle of superposition. The response of a linear system due to a sinusoidal input signal remains sinusoidal with an amplitude modification and phase shift, whereas a non-linear system produces distortion that gives rise to harmonic components of the input signal frequency in its output. In stochastic situations, the output of a linear system due to Gaussian random inputs preserves the Gaussian property and in nonlinear systems this is not the case. Linear systems are studied by a fairly general framework of techniques. Although there exist some methods to study nonlinear systems, there is no general methodology, which is universally applicable to nonlinear systems. For this reason nonlinear systems are in general quite complex.

Nonlinear differential equations are not easy to handle. System operation about an operating point is often very relevant in practice and an understanding of such a situation is provided by studying the model of the system obtained by linearizing it about the chosen operating point. The resulting linear model is studied with help of the well-established techniques for linear systems by considering deviations about the point and the related signals as sufficiently small.

Consider the problem of linearizing a function $f(x)$ about a point x_o. Expanding the function in Taylor series about the point

$$f(x)- f(x_o) = (df/dx) \Big|_{x = x_o} (x - x_o)$$

which is a linear relationship in the form:

$$\delta f(x) = m_o \, \delta x$$

The procedure for linearization for a general nonlinear equation in state space is as follows:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \quad (21)$$

where

$\mathbf{x}$ (t) = [x_1(t) ……..x_n(t)]^T

$\mathbf{u}$ (t) = [u_1(t) ……..u_r(t)]^T

and $\mathbf{f}(\mathbf{x},\mathbf{u})$ is a vector nonlinear function.

Linearization of this leads to the linear vector differential equation

$$\dot{\mathbf{x}}^*(t) = \mathbf{A}\mathbf{x}^*(t) + \mathbf{B}\mathbf{u}^*(t) \quad (22)$$

Where **A** and **B** are Jacobians containing the various partial derivative terms as follows:

$$\mathbf{A} = \begin{bmatrix} \frac{\partial f_1(\mathbf{x},\mathbf{u})}{\partial x_1} & \frac{\partial f_1(\mathbf{x},\mathbf{u})}{\partial x_n} \\ \frac{\partial f_n(\mathbf{x},\mathbf{u})}{\partial x_1} & \frac{\partial f_n(\mathbf{x},\mathbf{u})}{\partial x_n} \end{bmatrix}_{\substack{\mathbf{x}=\bar{\mathbf{x}} \\ \mathbf{u}=\bar{\mathbf{u}}}} \quad \mathbf{B} = \begin{bmatrix} \frac{\partial f_1(\mathbf{x},\mathbf{u})}{\partial u_1} & \frac{\partial f_1(\mathbf{x},\mathbf{u})}{\partial u_r} \\ \frac{\partial f_n(\mathbf{x},\mathbf{u})}{\partial u_1} & \frac{\partial f_n(\mathbf{x},\mathbf{u})}{\partial u_r} \end{bmatrix}_{\substack{\mathbf{x}=\bar{\mathbf{x}} \\ \mathbf{u}=\bar{\mathbf{u}}}} \qquad (23)$$

If the system parameters vary with time, it is called a time-varying (often referred to as time-variable or non stationary) system. For instance, a rocket represents a time-varying system as its mass changes with time during the course of its flight due to the expense of fuel.