3.4. Discrete-Time Systems or Sampled Data Systems

If a system variable (signal) *y*, at any arbitrary instant of time can be varied within known limits continuously, it is called "continuous". If a signal can take only known discrete amplitude values, then it is called a "quantized signal". If a signal is known only at certain discrete instants of time, then it is known as a discrete-time (or discrete) signal. If the signal values are given at uniformly sampled instants of time separated by an interval *T*, *T* is referred to as the sampling period. The signal itself is referred to as 'sampled'. Systems, in which such signals occur, are called discrete-time systems, or discrete systems or sampled-data systems. In general, if digital computers are employed in control systems, for instance to act as controllers, only quantized discrete-time data is processed. Linear time invariant discrete time systems are described by difference equations.

$$y(kT) + a_1 y(kT - T) + ... + a_{n-2} y(kT - nT + 2T) + a_{n-1} y(kT - nT + T) + a_n y(kT - nT$$
$$= b_0 u(kT) + b_1 u(kT - T) + ... + b_m u(kT - mT); k = 0,1,2,.....$$
(17)

They can be studied by applying the methods of *z*-transform, which for a discrete time signal *f(nT), n=0,1,2, …*, where *T* is the sampling interval, is defined as

$$F(z) = \sum_{n=0}^{\infty} f(nT) z^{-n} \quad (18)$$

z-transformation of Eq. (17) gives

$$\frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + ... + a_n z^{-n}} \quad (19)$$

Discrete time systems are described in state space as

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$$
$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k) \qquad (20)$$

Stability conditions for discrete time systems are discussed with reference to the *z*-plane. A linear time invariant discrete-time system described as above is stable if the eigenvalues of the A matrix (poles of the transfer function or roots of the characteristic polynomial) lie within the unit circle centered at the origin of the *z*-plane.

Strips parallel and symmetric to the real axis in the *s*-plane will have to be recognized due to sampling and the width of these strips is proportional to the sampling frequency *1/T*. The first of these is the primary strip that is of significance. The *z*-transform is related to the Laplace transform through the relation $z = e^{sT}$. According to this relation, the entire left half of the *s*-plane is transformed into the inside of the unit circle and the right half into the region outside the unit circle in the *z*-plane. The $j\omega$ axis winds itself into the unit circle itself with its various segments in the horizontal strips coinciding with the same circle periodically.

If continuous time systems are to be discretized, the minimum sampling frequency that is necessary to preserve the information in the sampled signal and to avoid aliasing effects is twice the highest frequency occurring in the signal spectrum. This criterion is called Shannon's sampling criterion. However, one rule of thumb in practice is to select *T* such that $\lambda_m T \leq 0.5$, where λ_m is the magnitude of the largest eigenvalue of the system. In actual practice it is desirable to make the sampling interval much smaller than the value specified by this rule. The result of such a choice is to force all poles to lie in a small lens shaped region in the *z*-plane as shown in Figure 10.

Fig.10: The region of normal operation in the z-plane