3.3. Linear Time-Invariant Systems

Laplace transformation applied to the state variable model of a general linear time invariant system with lumped parameters in the general state variable form

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$
$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du} \quad (13)$$

gives

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{AX}(s) + \mathbf{BU}(s)$$
$$\mathbf{Y}(s) = \mathbf{CX}(s) + \mathbf{DU}(s)$$

The Laplace transform of the vector of outputs is given by

$$\mathbf{Y}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s) \quad (14)$$

If we let **D=0** which is the common case,

$$\mathbf{Y}(s) = \frac{\mathbf{C}\ \mathrm{cof}(s\mathbf{I} - \mathbf{A})\mathbf{B}}{\det\ (s\mathbf{I} - \mathbf{A})}\mathbf{U}(s) \quad (15)$$

The denominator term on the right hand side of the above is the characteristic polynomial. The eigenvalues of the system matrix **A** are the poles. The coefficient of the term **U**(s) is the transfer function matrix whose *(i,j)*-th element happens to be transfer between the *i*-th input and the *j*-th output in the multi-input-multi-output system described by Eq. (15).

Solution of the state equation in time domain is direct by analogy with the first order scalar equation *dx/dt=ax+bu*.

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{Bu}(\tau)d\tau \quad (16)$$

The state variable representation is not unique; it depends on the choice of the set of state variables, which correspond, to the coordinate system in the n-dimensional space, which is referred to as the state space. Similarity transformation brings about a change in the state variable description without actually influencing the properties of the system. Certain state variable representations are termed as *canonical* because they involve minimal number of system parameters. State variable representation permits examination of additional properties such as *controllability* and *observability* of a system (see *System Characteristics*). Figure 9 illustrates these properties.

Fig.9: System controllability and observability with respect to segregated subsystems

Clearly, the controllable and observable part of the system is reflected in the input-output behavior and the transfer function of the overall system is given by the controllable and observable part only.

The state variable representation is a more complete description than the transfer function representation. It presents system behavior both internal and external while the transfer function gives the external (input-output) behavior only. The state space description is very appropriate for finite dimensional systems, that is, systems described by ordinary differential equations of a finite degree. If a system has time delays, the resulting delay differential equations cannot be represented easily in state space form. However, if a state space representation is desired, the delay terms have to be represented, in some sense of approximation, as finite dimensional elements. Thus the presence of delay terms in a differential equation gives rise to an arbitrary enlargement of the dimension of the state space.