Different classes of differential equations describe different types of dynamical systems. Lumped parameter systems are described by ordinary differential equations. Lumped linear continuous-time systems are described by linear differential equations. For instance, the n-th order linear differential equation with the single *input x* and single *output* y of the general form:

$$\sum_{i=0}^{n} a_i(t)\frac{d^i}{dt^i}y(t) = \sum_{i=0}^{m} b_i(t)\frac{d^i}{dt^i}x(t) \qquad (1)$$

satisfies the principle of superposition by virtue of its linear property. If the coefficients in the above are constant as in Eq. (2), it represents a linear time invariant system of the form:

$$\sum_{i=0}^{n} a_i\frac{d^i}{dt^i}y(t) = \sum_{i=0}^{m} b_i\frac{d^i}{dt^i}x(t) \qquad (2)$$

Working with linear time invariant systems becomes simplified with the help of Laplace transforms. The Lapalce transform of a function $f(t), t \in [0, \infty]$ is defined as

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt \qquad (3)$$

This transformation converts a linear differential equation into an algebraic form in the domain *s* that represents the complex frequency. Let $a_n = 1$ without loss of generality. Let the constant initial conditions be defined as

$$\left.\frac{d^k y}{dt^k}\right|_{t=0^+} \equiv y_0^k, k = 0,1,\ldots,n-1; \quad \left.\frac{d^k x}{dt^k}\right|_{t=0^+} \equiv x_0^k, k = 0,1,\ldots,m-1 \qquad (4)$$

Then the Laplace transform of Eq. (2) is given by

$$\sum_{i=0}^{n}[a_i(s^iY(s)-\sum_{k=0}^{i-1}s^{i-1-k}y_0^k)]=\sum_{i=0}^{m}[b_i(s^iX(s)-\sum_{k=0}^{i-1}s^{i-1-k}x_0^k)] \quad (5)$$

and the transform of the output is

$$Y(s)=\left[\frac{\sum_{i=0}^{m}b_is^i}{\sum_{i=0}^{n}a_is^i}\right]X(s)\quad+\quad\frac{\sum_{i=0}^{n}\sum_{k=0}^{i-1}a_is^{i-1-k}y_0^k}{\sum_{i=0}^{n}a_is^i}\quad-\quad\frac{\sum_{i=0}^{m}\sum_{k=0}^{i-1}b_is^{i-1-k}x_0^k}{\sum_{i=0}^{n}a_is^i} \quad (6)$$

The denominator term $\sum_{i=0}^{n}a_is^i$ is called the *characteristic polynomial*.

The response *Y(s)* consists of two components. The first term is due to the input and therefore it is referred to as the *forced response* or the *zero-state response*. The set of initial conditions (4) represents the initial *state* of the system. The coefficient of *X(s)* in the first term of the output expression is referred as the *transfer function* and it has to be obtained in the absence of initial conditions as the ratio of the Laplace transforms of the output to the input. The expression for output in Eq. (6) also contains another term that depends on the initial conditions only and not on the input. This component of the system output is known as the *free response* or the *zero-input response*.

If a unit impulse function or the Dirac delta function denoted as is considered as the input x(t) =δ *(t)*, *X(s)* =1, the forced response component in Eq. (6) happens to be equal to the transfer function itself. Thus, the transfer function may also be regarded as the (unit) impulse response and in time domain the unit impulse response is given by the inverse Laplace transform of the system transfer function. The impulse function is not an ordinary function of time. That is, the value of this function is not definitively defined at a given time. A unit impulse function δ *(t)* is indirectly defined by the following properties:

$$\int_0^{\infty} \delta(t)dt = 1 \quad (7)$$

and for any function $f(t), t \in [0, \infty]$ continuous at τ , defined in the ordinary sense

$$\int_0^{\infty} \delta(t-\tau) f(t)dt = f(\tau) \quad (8)$$

The inverse Laplace transform of Eq. (6) for *y(t)* is obtained be the method of partial fractions as follows.

Suppose that the characteristic polynomial has n_1 roots each equal to $-p_1$, n_2 roots each equal to $-p_2, \ldots, n_r$ roots each equal to $-p_r$ such that $\sum_{i=1}^{r} n_i = n$. Then $\sum_{i=0}^{n} a_i s^i = \prod_{i=1}^{r} (s+p_i)^{n_i}$ and the function $Y(s)/X(s)$ can be written as

$$F(s) = \frac{Y(s)}{X(s)} = b_n + \sum_{i=1}^{r} \sum_{k=1}^{n_i} \frac{c_{ik}}{(s+p_i)^k} \quad (9)$$

where b_n=0 unless $m=n$. The coefficients are given by

$$c_{ik} = \frac{1}{(n_i - k)!} \frac{d^{n_i-k}}{d^{n_i-k}} [(s+p_i)^{n_i} F(s)]\Big|_{s=-p_i} \quad (10)$$

These coefficients are also known as the residues of *F(s)* at $-p_i$, $i = 1, 2, \ldots r$. Inverse Laplace transformation of (9) gives:

$$f(t) = b_n \delta(t) + \sum_{i=1}^{r} \sum_{k=1}^{n_i} \frac{c_{ik}}{(k-1)!} t^{k-1} e^{-pt} \quad (11)$$

where $\delta\ (t)$ is the unit impulse function, and $b_n = 0$ unless $m=n$.

If a system does not contain dead time elements (delay elements) the transfer function *F(s)* is rational, that is, a ratio of two polynomials. The roots of the numerator polynomial are referred to as the *zeros* and the roots of the characteristic polynomial, or that of the denominator are termed the *poles* of the transfer function. These terms are suggestive of the nature of the function *F(s)* with reference to the complex frequency variable s. If *F(s)* is viewed as a potential function on the *s*-plane, the value of the function is zero at the *zeros*. At the points representing the zeros of the characteristic polynomial the value soars to infinity making the profile of the potential function *F(s)* at these points in the s-plane appear as poles. For this reason, the roots of the characteristic polynomial are called the poles. In the s-plane, a pole is shown as **x** and a zero as **o**.

The response as $t \rightarrow \infty$, is called the *steady state response*. The system response as a function of time before it reaches the steady state is called the *transient response*. The steady state value can be determined by applying the final value theorem:
$f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$, if the limit exists.

Notice that nature of *f(t)* depends on the values of p_i, the poles of the transfer function. When a pole is real the response component due to it is purely exponential. If it is negative, the response decays asymptotically in time and when it is positive, the response grows. Complex poles appear as conjugate pairs and the response due to such a pair is sinusoidal in nature. If the pair has a negative real part, the oscillations decay in time and when they have a positive real part, the oscillations grow in amplitude without limit. Referring to the complex s-plane, these conditions are interpreted as conditions for stability for linear time invariant dynamical systems in the following manner. If all the poles of the transfer function lie inside the left half of the *s*-plane, the system is asymptotically stable. If any pole lies on the imaginary axis, the system is critically stable and if any pole lies in the right half of the *s*-plane, then the system is unstable. These criteria are illustrated in Figure 8. Routh-Hurwitz stability criteria are used to detect the location of the roots, without actually solving the characteristic equation for its roots. Nyquist criterion ascertains the stability of a closed loop system by examining the transfer function of the open loop system. A more detailed discussion on the stability theory of dynamical systems is given elsewhere (see *Stability Concepts*).

Fig 8. Stability criteria for linear time invariant dynamic systems

The transfer function $F(j\omega)$ evaluated along the $j\omega$-axis of the s-plane is of significance as it represents the steady state response of the system to a sinusoidal input of frequency ω. It is a complex number with magnitude representing the amplification/attenuation and a phase angle that is the phase shift between the input and output signals.